

RANK PROPERTIES OF EXPOSED POSITIVE MAPS

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ABSTRACT. Let \mathcal{K} and \mathcal{H} be finite dimensional Hilbert spaces and let \mathfrak{P} denote the cone of all positive linear maps acting from $\mathfrak{B}(\mathcal{K})$ into $\mathfrak{B}(\mathcal{H})$. We show that each map of the form $\phi(X) = AXA^*$ or $\phi(X) = AX^T A^*$ is an exposed point of \mathfrak{P} .

1. INTRODUCTION

Let us start with setting up some notation and terminology. Assume V is a finite dimensional normed space. A subset C is a *convex cone* (or simply *cone*) in V if $\alpha x + \beta y \in C$ for all $x, y \in C$ and $\alpha, \beta \in \mathbb{R}_+$. A cone V is *pointed* if $V \cap (-V) = \{0\}$. A convex subcone $F \subset C$ is called a *face* if $x - y \in F$ implies $y \in F$ for every $y \in C$. Any face F such that $F \neq \{0\}$ and $F \neq C$ will be called *proper*. A proper face F is said to be *maximal* if for any face G such that $F \subset G \subset C$ we have either $G = F$ or $G = C$. If $K \subset C$ is any subset then by $F(K)$ we will denote the smallest face containing K . If $K = \{x\}$ for some $x \in C$ then we will write $F(x)$ instead of $F(\{x\})$. An element $x \in C$ will be called *extremal* if $F(x) = \overline{\mathbb{R}_+}x$, where $\overline{\mathbb{R}_+}$ is the set of all non-negative real numbers. The set of all extremal elements will be denoted by $\text{ext } C$. In the sequel we will need the following

Lemma 1.1. *If C is a cone and $F \subset C$ is a face then $\text{ext } F = F \cap \text{ext } C$.*

Proof. Let $x \in \text{ext } F$. We should show that $x \in \text{ext } C$. Assume that $y \in C$ and $x - y \in C$. Since $x \in F$ and F is a face, $y \in F$ and $x - y \in F$. Now, extremality of x in F implies $y = \lambda x$ for some nonnegative constant λ . Thus $x \in \text{ext } C$. We proved $\text{ext } F \subset F \cap \text{ext } C$. The converse inclusion is obvious. \square

Assume now that W is another finite dimensional normed space, and V and W are dual to each other with respect to bilinear pairing $\langle \cdot, \cdot \rangle_d$. For a subset $C \subset V$ (respectively $D \subset W$) we define the *dual cone* $C^\circ = \{y \in W : \langle x, y \rangle_d \geq 0 \text{ for all } x \in C\}$ (respectively $D^\circ = \{x \in V : \langle x, y \rangle_d \geq 0 \text{ for all } y \in D\}$). One can show that $C^{\circ\circ}$ is the smallest closed cone containing C .

Assume $C \subset V$ is a closed convex cone and $F \subset C$ is a face. We define $F' = \{y \in C^\circ : \langle x, y \rangle_d = 0 \text{ for all } x \in F\}$. It is clear that F' is a closed face of C° . We

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say that a face F of a closed convex cone C is *exposed* if there exists $y_0 \in C^\circ$ such that $F = \{x \in C : \langle x, y_0 \rangle_d = 0\}$. We have the following

Lemma 1.2 ([6]). *Let F be a closed face of a closed convex cone C . Then F is exposed if and only if $F = F''$. The set F'' is the smallest closed face containing F .*

An element $x \in C$ is called an *exposed point* of C if $\overline{\mathbb{R}_+ x}$ is an exposed face of C . The set of all exposed points of C will be denoted by $\exp C$.

Now, let \mathcal{K} and \mathcal{H} be finite dimensional Hilbert spaces and $\dim \mathcal{K} = m$, $\dim \mathcal{H} = n$. By $\mathfrak{B}(\mathcal{K})$ (respectively $\mathfrak{B}(\mathcal{H})$) we denote the C^* -algebra of all linear transformations acting on \mathcal{K} (respectively \mathcal{H}). Let $V = \mathfrak{B}(\mathfrak{B}(\mathcal{K}), \mathfrak{B}(\mathcal{H}))$ be the space of linear mappings from $\mathfrak{B}(\mathcal{K})$ into $\mathfrak{B}(\mathcal{H})$ and let $W = \mathfrak{B}(\mathcal{H}) \otimes \mathfrak{B}(\mathcal{K})$. Assume also that some antilinear selfadjoint involutions $\mathcal{K} \ni \xi \mapsto \bar{\xi} \in \mathcal{K}$ and $\mathcal{H} \ni \eta \mapsto \bar{\eta} \in \mathcal{H}$ are given. Following [17] (see also [6]) we define the following bilinear pairing between V and W

$$\langle \phi, X \otimes Y \rangle_d = \text{Tr}(\phi(Y)X^T) \quad (1)$$

where $\phi \in V$, $X \in \mathfrak{B}(\mathcal{H})$, $Y \in \mathfrak{B}(\mathcal{K})$ and T is the transposition on $\mathfrak{B}(\mathcal{H})$ determined by the given antilinear selfadjoint involution on \mathcal{H} .

Now, we choose some special convex cones $\mathfrak{P} \subset V$ and $\mathfrak{S} \subset W$. Namely \mathfrak{P} consists of positive maps, i.e. such maps ϕ that $\phi(\mathfrak{B}(\mathcal{K})_+) \subset \mathfrak{B}(\mathcal{H})_+$, while $\mathfrak{S} = \mathfrak{B}(\mathcal{H})_+ \otimes \mathfrak{B}(\mathcal{K})_+$ (its elements are sometimes called unnormalized separable states). It is known that these cones are dual to each other, i.e. $\mathfrak{S} = \mathfrak{P}^\circ$ (e.g. [10]).

The structure of extremal elements of \mathfrak{S} is simple. For two vectors x, y from a Hilbert space \mathcal{X} we let xy^* denote the rank 1 operator on \mathcal{X} such that $(xy^*)z = \langle y, z \rangle x$ for $z \in \mathcal{X}$. It follows from the definition of \mathfrak{S} that

$$\text{ext } \mathfrak{S} = \{\xi \xi^* \otimes \eta \eta^* : \xi \in \mathcal{H}, \eta \in \mathcal{K}\}.$$

The big challenge is to describe the structure of extremal elements of the cone \mathfrak{P} . The full description of the set $\text{ext } \mathfrak{P}$ is still not done. Only some partial results are known. In [16] extremal elements in the convex set of unital maps acting from $\mathfrak{B}(\mathbb{C}^2)$ into $\mathfrak{B}(\mathbb{C}^2)$ are characterized. As regards the full cone \mathfrak{P} of positive but not necessarily unital maps, it was proved in [20] that every maps of the form

$$\phi(X) = AXA^* \quad \text{or} \quad \phi(X) = AX^T A^*, \quad X \in \mathfrak{B}(\mathcal{K}), \quad (2)$$

($A \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$) are extremal in \mathfrak{P} . Moreover, several examples of non-decomposable extremal positive maps are scattered over the literature (see e.g. [2, 19, 13, 7, 1]).

Let us remind that due to Straszewicz's theorem ([18], see also [14]) the set $\exp \mathfrak{P}$ is dense in $\text{ext } \mathfrak{P}$. Thus, in order to do a full characterization of positive maps it is enough to describe fully the set of all exposed points of \mathfrak{P} . It was proved in [20] that if $\text{rank } A = 1$ or $\text{rank } A = m$ then a map of the form (2) is an exposed point of C . Recently, some new examples of exposed nondecomposable positive maps has appeared in the literature (see e.g. [8, 3, 4, 15, 5]).

The aim of this paper is to provide some new examples of exposed positive maps. We will use results concerning rank properties of extremal positive maps described in [12].

2. MAIN RESULT

Now we are ready to formulate our main theorem.

Theorem 2.1. *Every map of the form (2) is an exposed point of C .*

In the proof of the above theorem we will need more or less known two lemmas. For any $A \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ we denote $\|A\|_2 = (\text{Tr}(A^*A))^{1/2}$.

Lemma 2.2. *There is a unique linear map $\mathfrak{B}(\mathcal{K}, \mathcal{H}) \ni A \mapsto f_A \in (\mathcal{H} \otimes \mathcal{K})^*$ such that $f_A(\xi \otimes \eta) = \langle \bar{\xi}, A\eta \rangle$ for any $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$. Moreover, we have $\|f_A\| = \|A\|_2$ for $A \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$.*

Proof. Observe that $\mathcal{H} \times \mathcal{K} \ni (\xi, \eta) \mapsto \langle \bar{\xi}, A\eta \rangle \in \mathbb{C}$ is a bilinear form. It follows from the universality property of a tensor product that this form has a unique lift to a linear functional f_A on the $\mathcal{H} \otimes \mathcal{K}$. Now, if $f \in (\mathcal{H} \otimes \mathcal{K})^*$ then we define $\varphi(\xi, \eta) = f(\bar{\xi} \otimes \eta)$ for $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$. It is a sesquilinear form on $\mathcal{H} \times \mathcal{K}$, so there is $A \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ such that $\varphi(\xi, \eta) = \langle \xi, A\eta \rangle$. Hence we have $f(\xi \otimes \eta) = \varphi(\bar{\xi}, \eta) = \langle \bar{\xi}, A\eta \rangle = f_A(\xi \otimes \eta)$.

It remains to show the equality of norms. Let η_1, \dots, η_m be an orthonormal basis of \mathcal{K} and ξ_1, \dots, ξ_m be any system of vectors from \mathcal{H} . Observe that $\|\sum_i \xi_i \otimes \eta_i\|^2 = \sum_i \|\xi_i\|^2$ and $\|A\|_2^2 = \sum_i \|A\eta_i\|^2$. Thus

$$\begin{aligned} \left| f_A \left(\sum_i \xi_i \otimes \eta_i \right) \right| &= \left| \sum_i \langle \bar{\xi}_i, A\eta_i \rangle \right| \leq \sum_i \|\bar{\xi}_i\| \|A\eta_i\| \leq \\ &\leq \left(\sum_i \|\xi_i\|^2 \right)^{1/2} \left(\sum_i \|A\eta_i\|^2 \right)^{1/2} = \left\| \sum_i \xi_i \otimes \eta_i \right\| \|A\|_2. \end{aligned}$$

Hence $\|f_A\| \leq \|A\|_2$. Now, observe that $\|\sum_i \bar{A\eta_i} \otimes \eta_i\| = \left(\sum_i \|A\eta_i\|^2 \right)^{1/2} = \|A\|_2$ and $|f_A(\sum_i \bar{A\eta_i} \otimes \eta_i)| = \|A\|_2^2 = \|A\|_2 \|\sum_i \bar{A\eta_i} \otimes \eta_i\|$. Thus we conclude $\|f_A\| = \|A\|_2$. \square

Lemma 2.3. *Let $A \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ and $\text{rank} A \geq 2$. Assume that an operator $B \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$ satisfies the following condition: for any $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$ if $\langle \xi, A\eta \rangle = 0$ then $\langle \xi, B\eta \rangle = 0$. Then $B = 0$.*

Proof. Let $\eta_1, \eta_2, \dots, \eta_m$ be such that $\bar{\eta}_1, \bar{\eta}_2, \dots, \bar{\eta}_m$ form an orthonormal system of eigenvectors of the operator A^*A . Let $r = \text{rank} A$. Thus we may assume that $\bar{\eta}_{r+1}, \dots, \bar{\eta}_m$ correspond to zero eigenvalue while $\bar{\eta}_1, \dots, \bar{\eta}_r$ correspond to non-zero eigenvalues. Thus vectors $A\bar{\eta}_1, \dots, A\bar{\eta}_r$ span the image of the operator A . Take any $j \in \{r+1, \dots, m\}$. Observe that $A\bar{\eta}_j = 0$, so $\langle \xi, A\bar{\eta}_j \rangle = 0$ for each $\xi \in \mathcal{H}$. It follows from the assumption of the Lemma that $\langle \xi, B\eta_j \rangle = 0$ for any $\xi \in \mathcal{H}$, so $B\eta_j = 0$. Now, let $j \in \{1, \dots, r\}$. It follows from the assumption that for any $\xi \in A\mathcal{K}^\perp$ we have $\langle \xi, B\eta_j \rangle = 0$. Hence $B\eta_j \in A\mathcal{K}$. Now consider also some $k \in \{1, \dots, r\}$ such that $k \neq j$. For every $z \in \mathbb{C}$ define

$$\zeta_z = -\bar{z} \|A\bar{\eta}_k\|^2 A\bar{\eta}_j + \|A\bar{\eta}_j\|^2 A\bar{\eta}_k, \quad \rho_z = \bar{\eta}_j + z\bar{\eta}_k.$$

Observe that $\langle \zeta_z, A\rho_z \rangle = 0$. On the other hand we have

$$\begin{aligned} \langle \zeta_z, B\overline{\rho_z} \rangle &= -z\|A\overline{\eta_k}\|^2 \langle A\overline{\eta_j}, B\eta_j \rangle - |z|^2\|A\overline{\eta_k}\|^2 \langle A\overline{\eta_j}, B\eta_k \rangle + \\ &\quad + \|A\overline{\eta_j}\|^2 \langle A\overline{\eta_k}, B\eta_j \rangle + \overline{z}\|A\eta_j\|^2 \langle A\overline{\eta_k}, B\eta_k \rangle. \end{aligned}$$

It follows from the assumption that this expression is equal to zero for every $z \in \mathbb{C}$. Thus we conclude that $\langle A\overline{\eta_k}, B\eta_j \rangle = 0$ for any $k = 1, \dots, r$. Since $B\eta_j \in A\mathcal{K}$ and $A\overline{\eta_1}, \dots, A\overline{\eta_r}$ span $A\mathcal{K}$, we conclude that $B\eta_j = 0$.

Thus we proved that $B\eta_j = 0$ for any $j = 1, 2, \dots, m$. The Lemma follows from the fact that η_1, \dots, η_m is a basis of \mathcal{K} . \square

Proof of Theorem 2.1. Let us consider a map $\phi \in C$. It follows from Lemma 1.2 that ϕ is an exposed point if and only if $\{\phi\}'' = \overline{\mathbb{R}_+}\phi$. Let us calculate the face $\{\phi\}' \subset D$ firstly. Since any closed convex cone in finite dimensional space W is a closed convex hull of its extremal elements, in order to determine the face $\{\phi\}'$ it is enough to describe its extremal elements. They are those elements of $\{\phi\}'$ which are extremal in D (cf. Lemma 1.1). Thus, one should find all pairs $(\xi, \eta) \in \mathcal{H} \times \mathcal{K}$ such that $\xi\xi^* \otimes \eta\eta^* \in \{\phi\}'$. An easy calculation shows that this holds if and only if $\langle \overline{\xi}, \phi(\eta\eta^*)\overline{\xi} \rangle = 0$ or, equivalently, $\phi(\eta\eta^*)\overline{\xi} = 0$. As a consequence we get the following characterization of the face $\{\phi\}''$: if $\psi \in C$, then $\psi \in \{\phi\}''$ if and only if $\psi(\eta\eta^*)\overline{\xi} = 0$ for all pairs $(\xi, \eta) \in \mathcal{H} \times \mathcal{K}$ such that $\phi(\eta\eta^*)\overline{\xi} = 0$.

Now, assume that $\phi(X) = AXA^*$ where A is some linear map from \mathcal{K} into \mathcal{H} . One can easily show that $\xi\xi^* \otimes \eta\eta^* \in \{\phi\}'$ if and only if $\langle \overline{\xi}, A\eta \rangle = 0$. We will show that for $\psi \in C$ if $\psi \in \{\phi\}''$ then ψ is rank 1 non-increasing in the sense of [12]. Let $\eta \in \mathcal{K}$. Consider any $\xi \in \mathcal{H}$ such that $\xi \perp A\eta$. Then $\overline{\xi}\xi^* \otimes \eta\eta^* \in \{\phi\}'$ what is equivalent to $\phi(\eta\eta^*)\overline{\xi} = 0$. So, it follows from the preceding paragraph that $\psi(\eta\eta^*)\overline{\xi} = 0$ for any $\xi \in \{A\eta\}^\perp$. Thus $\psi(\eta\eta^*)$ is a non-negative multiple of rank 1 positive element $(A\eta)(A\eta)^*$.

Now, it follows from Theorem 2.2 in [12] that we have three possibilities:

- (i) $\psi(X) = \omega(X)Q$ for some positive functional ω on $\mathfrak{B}(\mathcal{K})$ and a 1-dimensional projection Q on \mathcal{H} ,
- (ii) $\psi(X) = BXB^*$ for some $B \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$,
- (iii) $\psi(X) = BXB^*$ for some $B \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$.

Assume firstly that ψ satisfies the condition (ii), i.e. $\psi(X) = BXB^*$ for some $B \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$. From the above considerations we conclude that B satisfies the following condition: for any $(\xi, \eta) \in \mathcal{H} \times \mathcal{K}$, if $\langle \overline{\xi}, A\eta \rangle = 0$ then $\langle \overline{\xi}, B\eta \rangle = 0$. Now apply Lemma 2.2. Using notations introduced there we can write the above condition as $\ker f_A \subset \ker f_B$. It is equivalent to the fact that f_B is a multiple of f_A . Hence $B = \lambda A$ for some $\lambda \in \mathbb{C}$, and $\psi = |\lambda|^2\phi$, and consequently $\psi \in \overline{\mathbb{R}_+}\phi$.

Secondly, consider the case (i), i.e. let $\psi(X) = \omega(X)Q$, where ω is some positive functional on $\mathfrak{B}(\mathcal{K})$ and Q is a 1-dimensional projection on \mathcal{H} . Let $\omega(X) = \text{Tr}(RX)$ where R is some positive operator on \mathcal{K} and let $Q = \zeta\zeta^*$ for some non-zero $\zeta \in \mathcal{H}$. We will show that $\text{rank} R \leq 1$. To this end observe that the condition $\psi \in \{\phi\}''$ is equivalent to the following: for any $(\xi, \eta) \in \mathcal{H} \times \mathcal{K}$, if $\langle \xi, A\eta \rangle = 0$ then $\langle \eta, R\eta \rangle = 0$ or $\langle \zeta, \xi \rangle = 0$. It follows that $\langle \eta, R\eta \rangle = 0$ provided that $A\eta = 0$, hence

$\ker A \subset \ker R$. Assume that there are two vectors $\eta_1, \eta_2 \in \mathcal{K}$ such that $R\eta_1, R\eta_2$ are linearly independent. Then $A\eta_1, A\eta_2$ are also linearly independent. Fix $i \in \{1, 2\}$. Since $\psi \in \{\phi\}''$, for any $\xi \in \{A\eta_i\}^\perp$ we have $\langle \eta_i, R\eta_i \rangle = 0$ or $\langle \zeta, \xi \rangle \zeta = 0$. We assumed $\langle \eta_i, R\eta_i \rangle \neq 0$, so we conclude that $\zeta \perp \xi$. Thus we proved that $\zeta \in \mathbb{C}A\eta_i$ for $i = 1, 2$. But $A\eta_1, A\eta_2$ are independent, so $\zeta = 0$ which is a contradiction to the assumption. Thus we proved that $R = \rho\rho^*$ for some $\rho \in \mathcal{K}$. Now, we can write $\phi(X) = \zeta\rho^*X\rho\zeta^* = \zeta\rho^*X(\zeta\rho^*)^*$ and we arrived at the previously described case (ii).

Finally, assume that $\phi(X) = CX^TC^*$ for some $C \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$. Observe that in this case the condition $\psi(\eta\eta^*)\bar{\xi} = 0$ is equivalent to $\langle \bar{\xi}, C\bar{\eta} \rangle = 0$. Thus $\psi \in \{\phi\}''$ if and only if for any $(\xi, \eta) \in \mathcal{H} \times \mathcal{K}$, $\langle \bar{\xi}, A\eta \rangle = 0$ implies $\langle \bar{\xi}, C\bar{\eta} \rangle = 0$. It follows from Lemma 2.3 that if $\text{rank} A \geq 2$ then $C = 0$. It remains to consider the case when $\text{rank} A \leq 1$. If $A\eta = 0$ then $C\bar{\eta} = 0$, hence $\ker A \subset \ker C$, and consequently $\text{rank} C \leq 1$. Then $C = \zeta\rho^*$ for some $\zeta \in \mathcal{H}$ and $\rho \in \mathcal{K}$. Hence $\psi(X) = \langle \rho, X^T\rho \rangle \zeta\zeta^*$. Since $X \mapsto \langle \rho, X^T\rho \rangle$ is a positive functional on $\mathfrak{B}(\mathcal{K})$, we came to the case (i). This ends the proof for $\phi(X) = AXA^*$.

If $\phi(X) = AX^TA^*$ then the proof is similar. To show that any $\psi \in \{\phi\}''$ is a multiple of ϕ one should firstly show that ψ is rank 1 non-increasing, then consider cases (iii), (i) and (ii). \square

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